Quantitative measure theory of Nilpotent groups

Romain Tessera

CNRS, Université Paris Cité et Sorbonne Université

Monday 3$^{\text{rd}}$ June, 2024

Quasi-isometry versus measure equivalence

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

Theorem (Gromov)

Two finitely generated groups Γ and Λ are **quasi-isometric** *if and only if there exists a* **topological coupling between them:**
a locally compact space X and **commuting, proper, cocompact** *actions of Λ and Γ by homeomorphisms on X.*

Quasi-isometry versus measure equivalence

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

Theorem (Gromov)

Two finitely generated groups Γ and Λ are **quasi-isometric** *if and only if there exists a* **topological coupling** *between them:*
a locally compact space X and **commuting, proper, cocompact** *actions of Λ and Γ by homeomorphisms on X.*

Definition (Gromov)

Two finitely generated groups Γ and Λ are **measure equivalent** if there exists a **measurable coupling** between them:
a σ-finite measure space X and **commuting, free, measure preserving** actions of Λ and Γ with **finite measure fundamental domains**.

Quasi-isometry versus measure equivalence

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

Theorem (Gromov)

*Two finitely generated groups Γ and Λ are **quasi-isometric** if and only if there exists a **topological coupling** between them:*
*a locally compact space X and **commuting, proper, cocompact** actions of Λ and Γ by homeomorphisms on X.*

Definition (Gromov)

Two finitely generated groups Γ and Λ are **measure equivalent** if there exists a **measurable coupling** between them:
a σ-finite measure space X and **commuting, free, measure preserving** actions of Λ and Γ with **finite measure fundamental domains**.

- fundamental example: Λ, Γ two lattices in a locally compact group G:
  $X = G$, Λ acting by left-multiplication, Γ by right-multiplication.

Quasi-isometry versus measure equivalence

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

Theorem (Gromov)

Two finitely generated groups Γ and Λ are **quasi-isometric** *if and only if there exists a* **topological coupling** *between them:*
a locally compact space X and **commuting, proper, cocompact** *actions of Λ and Γ by homeomorphisms on X.*

Definition (Gromov)

Two finitely generated groups Γ and Λ are **measure equivalent** if there exists a **measurable coupling** between them:
a σ-finite measure space X and **commuting, free, measure preserving** actions of Λ and Γ with **finite measure fundamental domains**.

- fundamental example: Λ, Γ two lattices in a locally compact group G:
  $X = G$, Λ acting by left-multiplication, Γ by right-multiplication.
- Any two infinite **amenable** groups are ME (Ornstein-Weiss 80).

- $\Lambda, \Gamma \curvearrowright X$ either topological or measurable coupling.

Measure equivalence cocycles

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

- $\Lambda, \Gamma \curvearrowright X$ either topological or measurable coupling.
- X_Γ, X_Λ fundamental domains for the actions of Γ and Λ.

Measure equivalence cocycles

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

- $\Lambda, \Gamma \curvearrowright X$ either topological or measurable coupling.
- X_Γ, X_Λ fundamental domains for the actions of Γ and Λ.

Definition (Cocycle)

Define $\alpha : \Lambda \times X_\Gamma \to \Gamma$ by:

Measure equivalence cocycles

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

- $\Lambda, \Gamma \curvearrowright X$ either topological or measurable coupling.
- X_Γ, X_Λ fundamental domains for the actions of Γ and Λ.

Definition (Cocycle)

Define $\alpha : \Lambda \times X_\Gamma \to \Gamma$ by: $\alpha(\lambda, x)$ is the unique $\gamma \in \Gamma$ such that

$$\gamma \lambda^{-1} x \in X_\Gamma,$$

(Similarly: $\beta : \Gamma \times X_\Lambda \to \Lambda$)

Measure equivalence cocycles

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

- $\Lambda, \Gamma \curvearrowright X$ either topological or measurable coupling.
- X_Γ, X_Λ fundamental domains for the actions of Γ and Λ.

Definition (Cocycle)

Define $\alpha : \Lambda \times X_\Gamma \to \Gamma$ by: $\alpha(\lambda, x)$ is the unique $\gamma \in \Gamma$ such that

$$\gamma \lambda^{-1} x \in X_\Gamma,$$

(Similarly: $\beta : \Gamma \times X_\Lambda \to \Lambda$)

- For a topological coupling, $\alpha(\cdot, x)$ is a **quasi-isometry** between Λ to Γ (uniform with respect to x).

Measure equivalence cocycles

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

- $\Lambda, \Gamma \curvearrowright X$ either topological or measurable coupling.
- X_Γ, X_Λ fundamental domains for the actions of Γ and Λ.

Definition (Cocycle)

Define $\alpha : \Lambda \times X_\Gamma \to \Gamma$ by: $\alpha(\lambda, x)$ is the unique $\gamma \in \Gamma$ such that

$$\gamma \lambda^{-1} x \in X_\Gamma,$$

(Similarly: $\beta : \Gamma \times X_\Lambda \to \Lambda$)

- For a topological coupling, $\alpha(\cdot, x)$ is a **quasi-isometry** between Λ to Γ (uniform with respect to x).
- **Problem:** for a measurable coupling, quantify how "distorted" the maps $\alpha(\cdot, x)$ are "in average".

Quantifying ME equivalence

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

Definition (L^p-measure equivalence)

Let $p \in (0, \infty]$. Two finitely generated groups Λ and Γ are L^p-ME if they are *ME*, and for all $s \in S_\Lambda$, $t \in S_\Gamma$, both

$$x \mapsto |\alpha(x, s)|_{S_\Gamma} \quad \text{and} \quad y \mapsto |\beta(y, t)|_{S_\Lambda}$$

are in $L^p(X_\Gamma)$ and $L^p(X_\Lambda)$.

Quantifying ME equivalence

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

Definition (L^p-measure equivalence)

Let $p \in (0, \infty]$. Two finitely generated groups Λ and Γ are L^p-ME if they are *ME*, and for all $s \in S_\Lambda$, $t \in S_\Gamma$, both

$$x \mapsto |\alpha(x, s)|_{S_\Gamma} \quad \text{and} \quad y \mapsto |\beta(y, t)|_{S_\Lambda}$$

are in $L^p(X_\Gamma)$ and $L^p(X_\Lambda)$.

- L^∞-ME: both $\alpha(x, \cdot) : \Lambda \to \Gamma$ and $\beta(y, \cdot) : \Gamma \to \Lambda$ are Lipschitz (uniformly with respect to x and y).

Definition (L^p-measure equivalence)

Let $p \in (0, \infty]$. Two finitely generated groups Λ and Γ are L^p-ME if they are *ME*, and for all $s \in S_\Lambda$, $t \in S_\Gamma$, both

$$x \mapsto |\alpha(x, s)|_{S_\Gamma} \quad \text{and} \quad y \mapsto |\beta(y, t)|_{S_\Lambda}$$

are in $L^p(X_\Gamma)$ and $L^p(X_\Lambda)$.

- L^∞-ME: both $\alpha(x, \cdot) : \Lambda \to \Gamma$ and $\beta(y, \cdot) : \Gamma \to \Lambda$ are Lipschitz (uniformly with respect to x and y).
- The **larger p**, the **stronger** the condition is.

Quantifying ME equivalence

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

Definition (L^p-measure equivalence)

Let $p \in (0, \infty]$. Two finitely generated groups Λ and Γ are L^p-ME if they are *ME*, and for all $s \in S_\Lambda$, $t \in S_\Gamma$, both

$$x \mapsto |\alpha(x,s)|_{S_\Gamma} \quad \text{and} \quad y \mapsto |\beta(y,t)|_{S_\Lambda}$$

are in $L^p(X_\Gamma)$ and $L^p(X_\Lambda)$.

- L^∞-ME: both $\alpha(x, \cdot) : \Lambda \to \Gamma$ and $\beta(y, \cdot) : \Gamma \to \Lambda$ are Lipschitz (uniformly with respect to x and y).
- The **larger** p, the **stronger** the condition is. For instance:

$$(L^\infty - ME) \Rightarrow (L^2 - ME)$$

Quantifying ME equivalence

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

> **Definition (L^p-measure equivalence)**
>
> Let $p \in (0, \infty]$. Two finitely generated groups Λ and Γ are L^p-ME if they are ME, and for all $s \in S_\Lambda$, $t \in S_\Gamma$, both
>
> $$x \mapsto |\alpha(x,s)|_{S_\Gamma} \quad \text{and} \quad y \mapsto |\beta(y,t)|_{S_\Lambda}$$
>
> are in $L^p(X_\Gamma)$ and $L^p(X_\Lambda)$.

- L^∞-ME: both $\alpha(x, \cdot) : \Lambda \to \Gamma$ and $\beta(y, \cdot) : \Gamma \to \Lambda$ are Lipschitz (uniformly with respect to x and y).
- The **larger** p, the **stronger** the condition is. For instance:

$$(L^\infty - ME) \Rightarrow (L^2 - ME) \Rightarrow (L^1 - ME)$$

Quantifying ME equivalence

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

Definition (L^p-measure equivalence)

Let $p \in (0, \infty]$. Two finitely generated groups Λ and Γ are L^p-ME if they are *ME*, and for all $s \in S_\Lambda$, $t \in S_\Gamma$, both

$$x \mapsto |\alpha(x, s)|_{S_\Gamma} \quad \text{and} \quad y \mapsto |\beta(y, t)|_{S_\Lambda}$$

are in $L^p(X_\Gamma)$ and $L^p(X_\Lambda)$.

- L^∞-ME: both $\alpha(x, \cdot) : \Lambda \to \Gamma$ and $\beta(y, \cdot) : \Gamma \to \Lambda$ are Lipschitz (uniformly with respect to x and y).

- The **larger** p, the **stronger** the condition is. For instance:

$$(L^\infty - ME) \Rightarrow (L^2 - ME) \Rightarrow (L^1 - ME) \Rightarrow (L^{1/2} - ME).$$

- (Shalom 04) if Λ and Γ are **amenable** and QI, then they are L^∞-ME.

Nilpotent groups

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

- **Lower central series**: $\Gamma_1 = \Gamma, \Gamma_2 = [\Gamma, \Gamma]$, $\Gamma_{i+1} = [\Gamma, \Gamma_i]$.

Nilpotent groups

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

- **Lower central series**: $\Gamma_1 = \Gamma, \Gamma_2 = [\Gamma, \Gamma]$, $\Gamma_{i+1} = [\Gamma, \Gamma_i]$.
- A group is m-**nilpotent** if $\Gamma_{m+1} = \{1\}$: abelian groups are 1-nilpotent, Heisenberg is 2-nilpotent...

Nilpotent groups

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

- **Lower central series**: $\Gamma_1 = \Gamma, \Gamma_2 = [\Gamma, \Gamma]$, $\Gamma_{i+1} = [\Gamma, \Gamma_i]$.
- A group is m-**nilpotent** if $\Gamma_{m+1} = \{1\}$: abelian groups are 1-nilpotent, Heisenberg is 2-nilpotent...
- **Malcev completion**: if Γ torsion free m-nilpotent, then it embeds as a uniform lattice in a simply connected m-nilpotent Lie group G.

Nilpotent groups

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

- **Lower central series**: $\Gamma_1 = \Gamma, \Gamma_2 = [\Gamma, \Gamma]$, $\Gamma_{i+1} = [\Gamma, \Gamma_i]$.
- A group is m-**nilpotent** if $\Gamma_{m+1} = \{1\}$: abelian groups are 1-nilpotent, Heisenberg is 2-nilpotent...
- **Malcev completion**: if Γ torsion free m-nilpotent, then it embeds as a uniform lattice in a simply connected m-nilpotent Lie group G.
- The **volume growth** of a simply connected nilpotent Lie group G is $V_G(r) \simeq r^D$, where $D = \sum_i i \dim A_i$, where $A_i = G_i/G_{i+1}$ (Bass-Guivarc'h).

Nilpotent groups

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

- **Lower central series**: $\Gamma_1 = \Gamma, \Gamma_2 = [\Gamma, \Gamma]$, $\Gamma_{i+1} = [\Gamma, \Gamma_i]$.
- A group is m-**nilpotent** if $\Gamma_{m+1} = \{1\}$: abelian groups are 1-nilpotent, Heisenberg is 2-nilpotent...
- **Malcev completion**: if Γ torsion free m-nilpotent, then it embeds as a uniform lattice in a simply connected m-nilpotent Lie group G.
- The **volume growth** of a simply connected nilpotent Lie group G is $V_G(r) \simeq r^D$, where $D = \sum_i i \dim A_i$, where $A_i = G_i/G_{i+1}$ (Bass-Guivarc'h).

Conjecture

Two simply connected nilpotent Lie groups are QI if and only if they are isomorphic.

Nilpotent groups

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

- **Lower central series**: $\Gamma_1 = \Gamma, \Gamma_2 = [\Gamma, \Gamma]$, $\Gamma_{i+1} = [\Gamma, \Gamma_i]$.
- A group is m-**nilpotent** if $\Gamma_{m+1} = \{1\}$: abelian groups are 1-nilpotent, Heisenberg is 2-nilpotent...
- **Malcev completion**: if Γ torsion free m-nilpotent, then it embeds as a uniform lattice in a simply connected m-nilpotent Lie group G.
- The **volume growth** of a simply connected nilpotent Lie group G is $V_G(r) \simeq r^D$, where $D = \sum_i i \dim A_i$, where $A_i = G_i/G_{i+1}$ (Bass-Guivarc'h).

Conjecture

Two simply connected nilpotent Lie groups are QI if and only if they are isomorphic.

Problem

Classify simply connected/ finitely generated nilpotent groups up to L^p-ME.

Growth function

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

Theorem (Bowen 16)

If Λ and Γ are L^1-ME, then $V_\Lambda \approx V_\Gamma$.

Growth function

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

Theorem (Bowen 16)

If Λ and Γ are L^1-ME, then $V_\Lambda \approx V_\Gamma$.

Theorem (Delabie-Koivisto-Le Maître-T 20/ Correia 24)

If $V_\Gamma(r) \simeq r^d$ and $V_\Lambda(r) \simeq r^{d+k}$, with $k \geq 1$ then Γ and Λ are not L^p-ME for $p \geq \frac{d}{d+k}$.

Growth function

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

Theorem (Bowen 16)

If Λ and Γ are L^1-ME, then $V_\Lambda \approx V_\Gamma$.

Theorem (Delabie-Koivisto-Le Maître-T 20/ Correia 24)

If $V_\Gamma(r) \simeq r^d$ and $V_\Lambda(r) \simeq r^{d+k}$, with $k \geq 1$ then Γ and Λ are not L^p-ME for $p \geq \frac{d}{d+k}$.

Growth function

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

Theorem (Bowen 16)

If Λ and Γ are L^1-ME, then $V_\Lambda \approx V_\Gamma$.

Theorem (Delabie-Koivisto-Le Maître-T 20/ Correia 24)

If $V_\Gamma(r) \simeq r^d$ and $V_\Lambda(r) \simeq r^{d+k}$, with $k \geq 1$ then Γ and Λ are not L^p-ME for $p \geq \frac{d}{d+k}$.

Example: $\mathbb{Z}$ and $\mathbb{Z}^2$ are not $L^{1/2}$-ME.

Growth function

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

Theorem (Bowen 16)

If Λ and Γ are L^1-ME, then $V_\Lambda \approx V_\Gamma$.

Theorem (Delabie-Koivisto-Le Maître-T 20/ Correia 24)

If $V_\Gamma(r) \simeq r^d$ and $V_\Lambda(r) \simeq r^{d+k}$, with $k \geq 1$ then Γ and Λ are not L^p-ME for $p \geq \frac{d}{d+k}$.

Example: $\mathbb{Z}$ and $\mathbb{Z}^2$ are not $L^{1/2}$-ME.

Theorem (Delabie-Llosa-T 24)

If $V_\Gamma(r) \simeq r^d$ and $V_\Lambda(r) \simeq r^{d+k}$, then Γ and Λ are L^p-ME for all $p < \frac{d}{d+k}$.

Growth function

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

Theorem (Bowen 16)

If Λ and Γ are L^1-ME, then $V_\Lambda \approx V_\Gamma$.

Theorem (Delabie-Koivisto-Le Maître-T 20/ Correia 24)

If $V_\Gamma(r) \simeq r^d$ and $V_\Lambda(r) \simeq r^{d+k}$, with $k \geq 1$ then Γ and Λ are not L^p-ME for $p \geq \frac{d}{d+k}$.

Example: $\mathbb{Z}$ and $\mathbb{Z}^2$ are not $L^{1/2}$-ME.

Theorem (Delabie-Llosa-T 24)

If $V_\Gamma(r) \simeq r^d$ and $V_\Lambda(r) \simeq r^{d+k}$, then Γ and Λ are L^p-ME for all $p < \frac{d}{d+k}$.

Example: $\mathbb{Z}$ and $\mathbb{Z}^2$ are L^p-ME for all $p < 1/2$.

Growth function

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

Theorem (Bowen 16)

If Λ and Γ are L^1-ME, then $V_\Lambda \approx V_\Gamma$.

Theorem (Delabie-Koivisto-Le Maître-T 20/ Correia 24)

If $V_\Gamma(r) \simeq r^d$ and $V_\Lambda(r) \simeq r^{d+k}$, with $k \geq 1$ then Γ and Λ are not L^p-ME for $p \geq \frac{d}{d+k}$.

Example: $\mathbb{Z}$ and $\mathbb{Z}^2$ are not $L^{1/2}$-ME.

Theorem (Delabie-Llosa-T 24)

If $V_\Gamma(r) \simeq r^d$ and $V_\Lambda(r) \simeq r^{d+k}$, then Γ and Λ are L^p-ME for all $p < \frac{d}{d+k}$.

Example: $\mathbb{Z}$ and $\mathbb{Z}^2$ are L^p-ME for all $p < 1/2$.

Corollary

Two finitely generated nilpotent groups are L^p-ME for all $p < 1$ if and only if they have same volume growth.

Growth function

Theorem (Bowen 16)

If Λ and Γ are L^1-ME, then $V_\Lambda \approx V_\Gamma$.

Theorem (Delabie-Koivisto-Le Maître-T 20/ Correia 24)

If $V_\Gamma(r) \simeq r^d$ and $V_\Lambda(r) \simeq r^{d+k}$, with $k \geq 1$ then Γ and Λ are not L^p-ME for $p \geq \frac{d}{d+k}$.

Example: $\mathbb{Z}$ and $\mathbb{Z}^2$ are not $L^{1/2}$-ME.

Theorem (Delabie-Llosa-T 24)

If $V_\Gamma(r) \simeq r^d$ and $V_\Lambda(r) \simeq r^{d+k}$, then Γ and Λ are L^p-ME for all $p < \frac{d}{d+k}$.

Example: $\mathbb{Z}$ and $\mathbb{Z}^2$ are L^p-ME for all $p < 1/2$.

Corollary

Two finitely generated nilpotent groups are L^p-ME for all $p < 1$ if and only if they have same volume growth.

Example: $\mathbb{Z}^4$ and $\mathrm{Heis}(\mathbb{Z})$ are L^p-ME for all $p < 1$.

Carnot graded nilpotent Lie groups

- We denote $\mathfrak{g}_i$ the lower central series of the Lie algebra $\mathfrak{g}$.

Carnot graded nilpotent Lie groups

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

- We denote $\mathfrak{g}_i$ the lower central series of the Lie algebra $\mathfrak{g}$.
- The lower central series gives rise to a **filtration** of $\mathfrak{g}$: $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$.

Carnot graded nilpotent Lie groups

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

- We denote $\mathfrak{g}_i$ the lower central series of the Lie algebra $\mathfrak{g}$.
- The lower central series gives rise to a **filtration** of $\mathfrak{g}$: $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$.
- A Lie algebra is called **Carnot gradable** if this filtration comes from a grading: $\mathfrak{g} = \bigoplus_i m_i$ satisfying $\mathfrak{g}_j = \bigoplus_{i \geqslant j} m_i$ and $[m_i, m_j] \subset m_{i+j}$.

Carnot graded nilpotent Lie groups

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

- We denote $\mathfrak{g}_i$ the lower central series of the Lie algebra $\mathfrak{g}$.
- The lower central series gives rise to a **filtration** of $\mathfrak{g}$: $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$.
- A Lie algebra is called **Carnot gradable** if this filtration comes from a grading: $\mathfrak{g} = \bigoplus_i m_i$ satisfying $\mathfrak{g}_j = \bigoplus_{i \geqslant j} m_i$ and $[m_i, m_j] \subset m_{i+j}$.
- It is always possible to associate a Carnot graded Lie algebra $\mathrm{gr}(\mathfrak{g})$ to any nilpotent Lie algebra $\mathfrak{g}$: $\mathrm{gr}(\mathfrak{g}) = \bigoplus_{i \geqslant 1} m_i$ for $m_i = \mathfrak{g}_i / \mathfrak{g}_{i+1}$.

Carnot graded nilpotent Lie groups

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

- We denote $\mathfrak{g}_i$ the lower central series of the Lie algebra $\mathfrak{g}$.
- The lower central series gives rise to a **filtration** of $\mathfrak{g}$: $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$.
- A Lie algebra is called **Carnot gradable** if this filtration comes from a grading: $\mathfrak{g} = \bigoplus_i m_i$ satisfying $\mathfrak{g}_j = \bigoplus_{i \geqslant j} m_i$ and $[m_i, m_j] \subset m_{i+j}$.
- It is always possible to associate a Carnot graded Lie algebra $\mathrm{gr}(\mathfrak{g})$ to any nilpotent Lie algebra $\mathfrak{g}$: $\mathrm{gr}(\mathfrak{g}) = \bigoplus_{i \geqslant 1} m_i$ for $m_i = \mathfrak{g}_i/\mathfrak{g}_{i+1}$.
- Observe that $\mathrm{gr}(G)$ has **same growth degree** as G.

Carnot graded nilpotent Lie groups

- We denote $\mathfrak{g}_i$ the lower central series of the Lie algebra $\mathfrak{g}$.
- The lower central series gives rise to a **filtration** of $\mathfrak{g}$: $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$.
- A Lie algebra is called **Carnot gradable** if this filtration comes from a grading: $\mathfrak{g} = \bigoplus_i m_i$ satisfying $\mathfrak{g}_j = \bigoplus_{i \geqslant j} m_i$ and $[m_i, m_j] \subset m_{i+j}$.
- It is always possible to associate a Carnot graded Lie algebra $\mathrm{gr}(\mathfrak{g})$ to any nilpotent Lie algebra $\mathfrak{g}$: $\mathrm{gr}(\mathfrak{g}) = \bigoplus_{i \geqslant 1} m_i$ for $m_i = \mathfrak{g}_i / \mathfrak{g}_{i+1}$.
- Observe that $\mathrm{gr}(G)$ has **same growth degree** as G.

Theorem (Pansu 89)

If two finitely generated nilpotent groups are QI, then they have isomorphic Carnot-graded associated Lie groups.

Carnot graded nilpotent Lie groups

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

- We denote $\mathfrak{g}_i$ the lower central series of the Lie algebra $\mathfrak{g}$.
- The lower central series gives rise to a **filtration** of $\mathfrak{g}$: $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$.
- A Lie algebra is called **Carnot gradable** if this filtration comes from a grading: $\mathfrak{g} = \bigoplus_i m_i$ satisfying $\mathfrak{g}_j = \bigoplus_{i \geqslant j} m_i$ and $[m_i, m_j] \subset m_{i+j}$.
- It is always possible to associate a Carnot graded Lie algebra $\mathrm{gr}(\mathfrak{g})$ to any nilpotent Lie algebra $\mathfrak{g}$: $\mathrm{gr}(\mathfrak{g}) = \bigoplus_{i \geqslant 1} m_i$ for $m_i = \mathfrak{g}_i/\mathfrak{g}_{i+1}$.
- Observe that $\mathrm{gr}(G)$ has **same growth degree** as G.

Theorem (Pansu 89)

If two finitely generated nilpotent groups are QI, then they have isomorphic Carnot-graded associated Lie groups.

Theorem (Austin 16)

If two finitely generated nilpotent groups are L^1-ME, then they have isomorphic Carnot-graded associated Lie groups.

Carnot graded nilpotent Lie groups

- We denote $\mathfrak{g}_i$ the lower central series of the Lie algebra $\mathfrak{g}$.
- The lower central series gives rise to a **filtration** of $\mathfrak{g}$: $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$.
- A Lie algebra is called **Carnot gradable** if this filtration comes from a grading: $\mathfrak{g} = \bigoplus_i m_i$ satisfying $\mathfrak{g}_j = \bigoplus_{i \geqslant j} m_i$ and $[m_i, m_j] \subset m_{i+j}$.
- It is always possible to associate a Carnot graded Lie algebra $\mathrm{gr}(\mathfrak{g})$ to any nilpotent Lie algebra $\mathfrak{g}$: $\mathrm{gr}(\mathfrak{g}) = \bigoplus_{i \geqslant 1} m_i$ for $m_i = \mathfrak{g}_i / \mathfrak{g}_{i+1}$.
- Observe that $\mathrm{gr}(G)$ has **same growth degree** as G.

Theorem (Pansu 89)

If two finitely generated nilpotent groups are QI, then they have isomorphic Carnot-graded associated Lie groups.

Theorem (Austin 16)

If two finitely generated nilpotent groups are L^1-ME, then they have isomorphic Carnot-graded associated Lie groups.

Classification up to L^1-ME

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

Theorem (Austin 16)

If two finitely generated nilpotent groups are L^1-ME, then they have isomorphic Carnot-graded associated Lie groups.

Classification up to L^1-ME

Theorem (Austin 16)

If two finitely generated nilpotent groups are L^1-ME, then they have isomorphic Carnot-graded associated Lie groups.

Example: $\mathbb{Z}^4$ and $\text{Heis}(\mathbb{Z})$ are not L^1-ME.

Classification up to L^1-ME

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

Theorem (Austin 16)

If two finitely generated nilpotent groups are L^1-ME, then they have isomorphic Carnot-graded associated Lie groups.

Example: $\mathbb{Z}^4$ and $\text{Heis}(\mathbb{Z})$ are not L^1-ME. Conversely:

Theorem (Delabie-Llosa-T 24)

If two finitely generated nilpotent groups have isomorphic Carnot-graded associated Lie groups, then they are L^p-ME for some (explicit) $p > 1$.

Classification up to L^1-ME

Theorem (Austin 16)

If two finitely generated nilpotent groups are L^1-ME, then they have isomorphic Carnot-graded associated Lie groups.

Example: $\mathbb{Z}^4$ and Heis$(\mathbb{Z})$ are not L^1-ME. Conversely:

Theorem (Delabie-Llosa-T 24)

If two finitely generated nilpotent groups have isomorphic Carnot-graded associated Lie groups, then they are L^p-ME for some (explicit) $p > 1$.

Corollary (Classification up to L^1-ME)

Two finitely generated nilpotent groups are L^1-ME if and only if they have isomorphic Carnot-graded associated Lie groups.

Classification up to L^1-ME

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

Theorem (Austin 16)

If two finitely generated nilpotent groups are L^1-ME, then they have isomorphic Carnot-graded associated Lie groups.

Example: $\mathbb{Z}^4$ and Heis($\mathbb{Z}$) are not L^1-ME. Conversely:

Theorem (Delabie-Llosa-T 24)

If two finitely generated nilpotent groups have isomorphic Carnot-graded associated Lie groups, then they are L^p-ME for some (explicit) $p > 1$.

Corollary (Classification up to L^1-ME)

Two finitely generated nilpotent groups are L^1-ME if and only if they have isomorphic Carnot-graded associated Lie groups.

Problem: Find the supremum over all p such that two nilpotent groups (with same Carnot-graded) are L^p-ME.

Concrete examples

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

Definition (filiform Lie algebra)

The (standard) filiform n-nilpotent Lie algebra $\mathfrak{f}_n$ is the Lie algebra of dimension n with basis $\{x_1, x_2, \ldots, x_n\}$ satisfying $[x_1, x_i] = x_{i+1}$ for $2 \leqslant i \leqslant n - 1$ and $[x_i, x_j] = 0$ for $1 < i \leqslant j \leqslant n$ or if $(i, j) = (1, n)$.

Concrete examples

Definition (filiform Lie algebra)

The (standard) filiform n-nilpotent Lie algebra $\mathfrak{l}_n$ is the Lie algebra of dimension n with basis $\{x_1, x_2, \ldots, x_n\}$ satisfying $[x_1, x_i] = x_{i+1}$ for $2 \leqslant i \leqslant n-1$ and $[x_i, x_j] = 0$ for $1 < i \leqslant j \leqslant n$ or if $(i, j) = (1, n)$.

Observe that L_n is $n-1$-step nilpotent, and L_3 is the Heisenberg group Heis.

Concrete examples

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

Definition (filiform Lie algebra)

The (standard) filiform n-nilpotent Lie algebra $\mathfrak{l}_n$ is the Lie algebra of dimension n with basis $\{x_1, x_2, \ldots, x_n\}$ satisfying $[x_1, x_i] = x_{i+1}$ for $2 \leqslant i \leqslant n-1$ and $[x_i, x_j] = 0$ for $1 < i \leqslant j \leqslant n$ or if $(i, j) = (1, n)$.

Observe that L_n is $n-1$-step nilpotent, and L_3 is the Heisenberg group Heis.

Definition (Central products)

We denote $G_{m,n}$ the central product of L_m with L_n obtained by identifying x_m with x_n (in the respective Lie algebras).

Concrete examples

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

Definition (filiform Lie algebra)

The (standard) filiform n-nilpotent Lie algebra $\mathfrak{l}_n$ is the Lie algebra of dimension n with basis $\{x_1, x_2, \ldots, x_n\}$ satisfying $[x_1, x_i] = x_{i+1}$ for $2 \leqslant i \leqslant n - 1$ and $[x_i, x_j] = 0$ for $1 < i \leqslant j \leqslant n$ or if $(i, j) = (1, n)$.

Observe that L_n is $n - 1$-step nilpotent, and L_3 is the Heisenberg group Heis.

Definition (Central products)

We denote $G_{m,n}$ the central product of L_m with L_n obtained by identifying x_m with x_n (in the respective Lie algebras).

Concrete examples

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

Definition (filiform Lie algebra)

The (standard) filiform n-nilpotent Lie algebra $\mathfrak{l}_n$ is the Lie algebra of dimension n with basis $\{x_1, x_2, \ldots, x_n\}$ satisfying $[x_1, x_i] = x_{i+1}$ for $2 \leqslant i \leqslant n-1$ and $[x_i, x_j] = 0$ for $1 < i \leqslant j \leqslant n$ or if $(i, j) = (1, n)$.

Observe that L_n is $n-1$-step nilpotent, and L_3 is the Heisenberg group Heis.

Definition (Central products)

We denote $G_{m,n}$ the central product of L_m with L_n obtained by identifying x_m with x_n (in the respective Lie algebras).

If $m > n$, then $\mathrm{gr}(G_{m,n}) = L_m \times L_{n-1}$. For instance $\mathrm{gr}(G_{4,3}) = \mathrm{Heis} \times \mathbb{R}^2$.

Concrete examples

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

Definition (filiform Lie algebra)

The (standard) filiform n-nilpotent Lie algebra $\mathfrak{l}_n$ is the Lie algebra of dimension n with basis $\{x_1, x_2, \ldots, x_n\}$ satisfying $[x_1, x_i] = x_{i+1}$ for $2 \leqslant i \leqslant n-1$ and $[x_i, x_j] = 0$ for $1 < i < j \leqslant n$ or if $(i,j) = (1,n)$.

Observe that L_n is $n-1$-step nilpotent, and L_3 is the Heisenberg group Heis.

Definition (Central products)

We denote $G_{m,n}$ the central product of L_m with L_n obtained by identifying x_m with x_n (in the respective Lie algebras).

If $m > n$, then $\mathrm{gr}(G_{m,n}) = L_m \times L_{n-1}$. For instance $\mathrm{gr}(G_{4,3}) = \mathrm{Heis} \times \mathbb{R}^2$.

Theorem (Delabie-Llosa-T 24)

For all $m > n \geq 3$ and all $p < \frac{m-1}{n-1}$, the groups $G_{m,n}$ and $\mathrm{gr}(G_{m,n})$ are L^p-ME.

Concrete examples

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

Definition (filiform Lie algebra)

The (standard) filiform n-nilpotent Lie algebra $\mathfrak{l}_n$ is the Lie algebra of dimension n with basis $\{x_1, x_2, \ldots, x_n\}$ satisfying $[x_1, x_i] = x_{i+1}$ for $2 \leqslant i \leqslant n-1$ and $[x_i, x_j] = 0$ for $1 < i < j \leqslant n$ or if $(i, j) = (1, n)$.

Observe that L_n is $n-1$-step nilpotent, and L_3 is the Heisenberg group Heis.

Definition (Central products)

We denote $G_{m,n}$ the central product of L_m with L_n obtained by identifying x_m with x_n (in the respective Lie algebras).

If $m > n$, then $\mathrm{gr}(G_{m,n}) = L_m \times L_{n-1}$. For instance $\mathrm{gr}(G_{4,3}) = \mathrm{Heis} \times \mathbb{R}^2$.

Theorem (Delabie-Llosa-T 24)

For all $m > n \geq 3$ and all $p < \frac{m-1}{n-1}$, the groups $G_{m,n}$ and $\mathrm{gr}(G_{m,n})$ are L^p-ME.

The critical value $p_c = \frac{m-1}{n-1}$ is expected to be sharp.

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

Definition (Orbit equivalence)

Two free pmp actions $\Lambda \curvearrowright (Z, \mu)$ and $\Gamma \curvearrowright (Y, \nu)$ are **orbit equivalent** (OE), if there exists an isomorphism of measure spaces $\Psi : (Z, \mu) \to (Y, \nu)$ such that for a.e. $z \in Z$,

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

Definition (Orbit equivalence)

Two free pmp actions $\Lambda \curvearrowright (Z, \mu)$ and $\Gamma \curvearrowright (Y, \nu)$ are **orbit equivalent** (OE), if there exists an isomorphism of measure spaces $\Psi : (Z, \mu) \to (Y, \nu)$ such that for a.e. $z \in Z$,

$$\Psi (\Lambda \cdot z) = \Gamma \cdot \Psi(z).$$

Orbit equivalence versus Measure equivalence

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

Definition (Orbit equivalence)

Two free pmp actions $\Lambda \curvearrowright (Z, \mu)$ and $\Gamma \curvearrowright (Y, \nu)$ are **orbit equivalent** (OE), if there exists an isomorphism of measure spaces $\Psi : (Z, \mu) \to (Y, \nu)$ such that for a.e. $z \in Z$,

$$\Psi(\Lambda \cdot z) = \Gamma \cdot \Psi(z).$$

Proposition

Assume $\Gamma, \Lambda \curvearrowright Z$ *free pmp actions* **with same orbits**. *Then*

$$X = \{(z, y) \in Z \times Z \mid z \in \Gamma \cdot y\}$$

comes with a σ-finite measure (identifying it with $Z \times \Gamma$), and free commuting, measure-preserving Λ and Γ actions with $X_\Lambda = X_\Gamma = \{(z, z) \mid z \in Z\}$ as common fundamental domain.

Orbit equivalence versus Measure equivalence

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

Definition (Orbit equivalence)

Two free pmp actions $\Lambda \curvearrowright (Z, \mu)$ and $\Gamma \curvearrowright (Y, \nu)$ are **orbit equivalent** (OE), if there exists an isomorphism of measure spaces $\Psi : (Z, \mu) \to (Y, \nu)$ such that for a.e. $z \in Z$,

$$\Psi (\Lambda \cdot z) = \Gamma \cdot \Psi(z).$$

Proposition

Assume $\Gamma, \Lambda \curvearrowright Z$ free pmp actions **with same orbits**. *Then*

$$X = \{(z, y) \in Z \times Z \mid z \in \Gamma \cdot y\}$$

comes with a σ-finite measure (identifying it with $Z \times \Gamma$), and free commuting, measure-preserving Λ and Γ actions with $X_\Lambda = X_\Gamma = \{(z, z) \mid z \in Z\}$ as common fundamental domain. In particular Λ and Γ are **ME**.

Constructions of OE pmp actions

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

More precise results:

Theorem (Delabie-Llosa-T 24)

If $V_\Gamma(r) \simeq r^d$ and $V_\Lambda(r) \simeq r^{d+k}$ for $d, k \in \mathbb{N}$, then Γ and Λ have (profinite) pmp actions which are L^p-OE for all $p < \frac{d}{d+k}$.

Constructions of OE pmp actions

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

More precise results:

Theorem (Delabie-Llosa-T 24)

If $V_\Gamma(r) \simeq r^d$ and $V_\Lambda(r) \simeq r^{d+k}$ for $d, k \in \mathbb{N}$, then Γ and Λ have (profinite) pmp actions which are L^p-OE for all $p < \frac{d}{d+k}$.

Theorem (Delabie-Llosa-T 24)

If two finitely generated nilpotent groups have isomorphic Carnot-graded associated Lie groups, then they (profinite) pmp actions which are L^p-OE for some (explicit) $p > 1$.

Constructions of OE pmp actions

More precise results:

Theorem (Delabie-Llosa-T 24)

If $V_\Gamma(r) \simeq r^d$ and $V_\Lambda(r) \simeq r^{d+k}$ for $d, k \in \mathbb{N}$, then Γ and Λ have (profinite) pmp actions which are L^p-OE for all $p < \frac{d}{d+k}$.

Theorem (Delabie-Llosa-T 24)

If two finitely generated nilpotent groups have isomorphic Carnot-graded associated Lie groups, then they (profinite) pmp actions which are L^p-OE for some (explicit) $p > 1$.

Preliminaries:

- **The** 2-**odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows.

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

Preliminaries:

- **The** 2-**odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows. The generator a of $\mathbb{Z}$ acts as:
  $a \cdot (0,0,0,1,\ldots) = (1,0,0,1\ldots)$ (no carry)

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Preliminaries:

- **The** 2-**odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows. The generator a of $\mathbb{Z}$ acts as:
  $a \cdot (0,0,0,1,\ldots) = (1,0,0,1\ldots)$ (no carry)
  $a \cdot (1,0,0,\ldots) = (0,1,0,\ldots)$ (carry)

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Preliminaries:

- **The** 2-**odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows. The generator a of $\mathbb{Z}$ acts as:
  $a \cdot (0,0,0,1,\ldots) = (1,0,0,1\ldots)$ (no carry)
  $a \cdot (1,0,0,\ldots) = (0,1,0,\ldots)$ (carry)
  $a \cdot (1,1,1,0,\ldots) =$

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Preliminaries:

- **The** 2-**odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as
  follows. The generator a of $\mathbb{Z}$ acts as:
  $a \cdot (0,0,0,1,\ldots) = (1,0,0,1\ldots)$ (no carry)
  $a \cdot (1,0,0,\ldots) = (0,1,0,\ldots)$ (carry)
  $a \cdot (1,1,1,0,\ldots) = (0,0,0,1,\ldots)$ (carry cascade)

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

Preliminaries:

- **The** 2-**odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows. The generator a of $\mathbb{Z}$ acts as:
  $a \cdot (0,0,0,1,\ldots) = (1,0,0,1\ldots)$ (no carry)
  $a \cdot (1,0,0,\ldots) = (0,1,0,\ldots)$ (carry)
  $a \cdot (1,1,1,0,\ldots) = (0,0,0,1,\ldots)$ (carry cascade)

- **The** 4-**odometer:** : consider the action of $\mathbb{Z}$ on the $\{0,1,2,3\}^{\mathbb{N}}$, defined as follows. $a \cdot (1,2,0,3,\ldots) =$

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Preliminaries:

- **The** 2-**odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows. The generator a of $\mathbb{Z}$ acts as:
  $a \cdot (0,0,0,1,\ldots) = (1,0,0,1\ldots)$ (no carry)
  $a \cdot (1,0,0,\ldots) = (0,1,0,\ldots)$ (carry)
  $a \cdot (1,1,1,0,\ldots) = (0,0,0,1,\ldots)$ (carry cascade)
- **The** 4-**odometer:** : consider the action of $\mathbb{Z}$ on the $\{0,1,2,3\}^{\mathbb{N}}$, defined as follows. $a \cdot (1,2,0,3,\ldots) = (2,2,0,3,\ldots)$ (no carry)

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

Preliminaries:

- **The** 2-**odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows. The generator a of $\mathbb{Z}$ acts as:
  $a \cdot (0,0,0,1,\ldots) = (1,0,0,1\ldots)$ (no carry)
  $a \cdot (1,0,0,\ldots) = (0,1,0,\ldots)$ (carry)
  $a \cdot (1,1,1,0,\ldots) = (0,0,0,1,\ldots)$ (carry cascade)

- **The** 4-**odometer:** : consider the action of $\mathbb{Z}$ on the $\{0,1,2,3\}^{\mathbb{N}}$, defined as follows. $a \cdot (1,2,0,3,\ldots) = (2,2,0,3,\ldots)$ (no carry)
  $a \cdot (3,1,2,0,\ldots) =$

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

Preliminaries:

- **The 2-odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows. The generator a of $\mathbb{Z}$ acts as:
  $a \cdot (0,0,0,1,\ldots) = (1,0,0,1\ldots)$ (no carry)
  $a \cdot (1,0,0,\ldots) = (0,1,0,\ldots)$ (carry)
  $a \cdot (1,1,1,0,\ldots) = (0,0,0,1,\ldots)$ (carry cascade)

- **The 4-odometer:** : consider the action of $\mathbb{Z}$ on the $\{0,1,2,3\}^{\mathbb{N}}$, defined as follows. $a \cdot (1,2,0,3,\ldots) = (2,2,0,3,\ldots)$ (no carry)
  $a \cdot (3,1,2,0,\ldots) = (0,2,2,0,\ldots)$ (carry)

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

Preliminaries:

- **The** 2-**odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows. The generator a of $\mathbb{Z}$ acts as:
  $a \cdot (0,0,0,1,\ldots) = (1,0,0,1\ldots)$ (no carry)
  $a \cdot (1,0,0,\ldots) = (0,1,0,\ldots)$ (carry)
  $a \cdot (1,1,1,0,\ldots) = (0,0,0,1,\ldots)$ (carry cascade)

- **The** 4-**odometer:** : consider the action of $\mathbb{Z}$ on the $\{0,1,2,3\}^{\mathbb{N}}$, defined as follows. $a \cdot (1,2,0,3,\ldots) = (2,2,0,3,\ldots)$ (no carry)
  $a \cdot (3,1,2,0,\ldots) = (0,2,2,0,\ldots)$ (carry)
  $a \cdot (3,3,3,3,1,0,\ldots) =$

Preliminaries:

- **The** 2-**odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows. The generator a of $\mathbb{Z}$ acts as:
  $a \cdot (0,0,0,1,\ldots) = (1,0,0,1\ldots)$ (no carry)
  $a \cdot (1,0,0,\ldots) = (0,1,0,\ldots)$ (carry)
  $a \cdot (1,1,1,0,\ldots) = (0,0,0,1,\ldots)$ (carry cascade)

- **The** 4-**odometer:** : consider the action of $\mathbb{Z}$ on the $\{0,1,2,3\}^{\mathbb{N}}$, defined as follows. $a \cdot (1,2,0,3,\ldots) = (2,2,0,3,\ldots)$ (no carry)
  $a \cdot (3,1,2,0,\ldots) = (0,2,2,0,\ldots)$ (carry)
  $a \cdot (3,3,3,3,1,0,\ldots) = (0,0,0,0,2,0,\ldots)$ (carry cascade)

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

Preliminaries:

- **The 2-odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows. The generator a of $\mathbb{Z}$ acts as:
  $a \cdot (0,0,0,1,\ldots) = (1,0,0,1\ldots)$ (no carry)
  $a \cdot (1,0,0,\ldots) = (0,1,0,\ldots)$ (carry)
  $a \cdot (1,1,1,0,\ldots) = (0,0,0,1,\ldots)$ (carry cascade)

- **The 4-odometer:** : consider the action of $\mathbb{Z}$ on the $\{0,1,2,3\}^{\mathbb{N}}$, defined as follows. $a \cdot (1,2,0,3,\ldots) = (2,2,0,3,\ldots)$ (no carry)
  $a \cdot (3,1,2,0,\ldots) = (0,2,2,0,\ldots)$ (carry)
  $a \cdot (3,3,3,3,1,0,\ldots) = (0,0,0,0,2,0,\ldots)$ (carry cascade)

These actions **preserve the product measures** on $\{0,1\}^{\mathbb{N}}$ and $\{0,1,2,3\}^{\mathbb{N}}$.

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

Preliminaries:

- **The 2-odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows. The generator a of $\mathbb{Z}$ acts as:
  $a \cdot (0,0,0,1,\ldots) = (1,0,0,1\ldots)$ (no carry)
  $a \cdot (1,0,0,\ldots) = (0,1,0,\ldots)$ (carry)
  $a \cdot (1,1,1,0,\ldots) = (0,0,0,1,\ldots)$ (carry cascade)

- **The 4-odometer:** : consider the action of $\mathbb{Z}$ on the $\{0,1,2,3\}^{\mathbb{N}}$, defined as follows. $a \cdot (1,2,0,3,\ldots) = (2,2,0,3,\ldots)$ (no carry)
  $a \cdot (3,1,2,0,\ldots) = (0,2,2,0,\ldots)$ (carry)
  $a \cdot (3,3,3,3,1,0,\ldots) = (0,0,0,0,2,0,\ldots)$ (carry cascade)

These actions **preserve the product measures** on $\{0,1\}^{\mathbb{N}}$ and $\{0,1,2,3\}^{\mathbb{N}}$.

Two sequences belong to the **same orbit** if and only if they **differ by at most finitely many coordinates**.

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

The actions of $\mathbb{Z}$ and $\mathbb{Z}^2$:

- We let $\mathbb{Z}$ acts on the 4-odometer: $\{0, 1, 2, 3\}^{\mathbb{N}}$

The actions of $\mathbb{Z}$ and $\mathbb{Z}^2$:

- We let $\mathbb{Z}$ acts on the 4-odometer: $\{0, 1, 2, 3\}^{\mathbb{N}}$
- We let $\mathbb{Z}^2$ acts on a product of 2-odometers: $\{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$.

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Quantitative
measure
theory of
Nilpotent
groups

Romain
Tessera

The actions of $\mathbb{Z}$ and $\mathbb{Z}^2$:

- We let $\mathbb{Z}$ acts on the 4-odometer: $\{0, 1, 2, 3\}^{\mathbb{N}}$
- We let $\mathbb{Z}^2$ acts on a product of 2-odometers: $\{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$.

The orbit equivalence: $F : \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}} \to \{0, 1, 2, 3\}^{\mathbb{N}}$ is defined

$$F(x, y) = x + 2y.$$

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

The actions of $\mathbb{Z}$ and $\mathbb{Z}^2$:

- We let $\mathbb{Z}$ acts on the 4-odometer: $\{0,1,2,3\}^{\mathbb{N}}$
- We let $\mathbb{Z}^2$ acts on a product of 2-odometers: $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$.

The orbit equivalence: $F : \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \to \{0,1,2,3\}^{\mathbb{N}}$ is defined

$$F(x,y) = x + 2y.$$

Example: if $x = (0,1,1,\ldots)$, $y = (1,0,1,\ldots)$, then

$$F(x,y) = (0+2, 1+0, 1+2, \ldots) = (2,1,3,\ldots).$$

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

The actions of $\mathbb{Z}$ and $\mathbb{Z}^2$:

- We let $\mathbb{Z}$ acts on the 4-odometer: $\{0, 1, 2, 3\}^{\mathbb{N}}$
- We let $\mathbb{Z}^2$ acts on a product of 2-odometers: $\{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$.

The orbit equivalence: $F : \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}} \to \{0, 1, 2, 3\}^{\mathbb{N}}$ is defined

$$F(x, y) = x + 2y.$$

Example: if $x = (0, 1, 1, \ldots)$, $y = (1, 0, 1, \ldots)$, then

$$F(x, y) = (0 + 2, 1 + 0, 1 + 2, \ldots) = (2, 1, 3, \ldots).$$

Integrability: better from $\mathbb{Z}$ to $\mathbb{Z}^2$ than from $\mathbb{Z}^2$ to $\mathbb{Z}$, because **large carry cascades** happen more often on the $\mathbb{Z}^2$-action than on the $\mathbb{Z}$-action.